Complex interpolation of spaces of operators on ℓ_1

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Abstract

Within the theory of complex interpolation and θ-Hilbert spaces we extend classical results of Kwapień on absolutely $(r,1)$-summing operators on ℓ_1 with values in ℓ_p as well as their natural extensions for mixing operators invented by Maurey. Furthermore, we show that for $1 < p < 2$ every operator T on ℓ_1 with values in θ-type 2 spaces, $\theta = 2/p'$, is Rademacher p-summing. This is another extension of Kwapień's results, and by an extrapolation procedure a natural supplement to a statement of Pisier.

0 Introduction

Kwapień in [Kwa68] showed that for $1 \leq p \leq \infty$ and $1 \leq r \leq 2$ defined by $1/r = 1 - |1/2 - 1/p|$ every continuous and linear operator on ℓ_1 with values in ℓ_p is $(r,1)$-summing, i. e., maps unconditionally summable into absolutely r-summable sequences, and Pisier in [Pi79] proved that this result also holds whenever ℓ_p $(1 \leq p \leq 2)$ is replaced by an arbitrary p-convex and p'-concave Banach function space. Carl and the first author in [CD92] gave a generalization of Kwapień's result within the framework of mixing operators: For $2 \leq s \leq \infty$ such that $1/s = |1/2 - 1/p|$ every operator $T : \ell_1 \to \ell_p$ is $(s,1)$-mixing, i. e. the composition of T with an arbitrary s-summing operator on ℓ_p is 1-summing. While Kwapień and Pisier used interpolation techniques (e. g. the Three-Lines-Theorem together with results of Orlicz ($p = 1$ or $p = \infty$) and Grothendieck ($p = 2$)), Carl and Defant used a certain tensor product trick.

In this paper we suggest a systematic approach to all these results within the framework of complex interpolation and θ-Hilbert spaces which for example allows to replace ℓ_p in Kwapień's result by the Schatten class $\mathcal{S}_p$ and also covers the well-known results of Mitiagin [Kwa68] and [CD92] on the coincidence of summing/mixing operators on ℓ_2 and Schatten classes. Furthermore, we show that for $1 \leq p \leq 2$ every operator on ℓ_1 with values in a p-convex Banach function space X with finite concavity is Rademacher p-summing; in other terms, each operator $T : \ell_1 \to X$ maps weakly p-summable sequences into almost unconditionally convergent sequences. This result follows by extra- and interpolation between the two border cases $p = 1$ (trivial case) and $p = 2$ (X has type 2 if and only if X is 2-concave and has non-trivial concavity if and only if every operator $T : \ell_1 \to X$ is Rademacher 2-summing). Moreover, it is a natural supplement to the above mentioned result of Pisier—

note that in contrast to his result our result on Rademacher p-summing operators does not depend on the exact degree of concavity of the image space.

Finally, we refer the reader to a recent paper by Cerdà and Mastyło [CM99] where an extension of Kwapień's result within the framework of Calderón–Lozanovskiĭ spaces, in particular Orlicz spaces, is given.

We use standard notation and notions from Banach space theory, as presented e. g. in [LT77] and [LT79]. For $1 \leq p \leq \infty$ the number p' is defined by $1/p + 1/p' = 1$. If E is a Banach space, then B_E is its (closed) unit ball and E' its dual. As usual $\mathcal{L}(E, F)$ denotes the Banach space of all (bounded and linear) operators from E into F endowed with the operator norm $\|\cdot\|$. For $1 \leq p \leq 2 \leq q < \infty$ we denote by $\mathbf{T_p}(E)$ and $\mathbf{C_q}(E)$ the (Rademacher) type p constant and the (Rademacher) cotype q constant of a Banach space with these properties.

Let (Ω, Σ, μ) be a σ-finite and complete measure space, and denote all μ-a.e. equivalence classes of real-valued measurable functions on Ω by $L_0(\mu)$. A Banach space $X = X(\mu)$ of (equivalence classes of) functions in $L_0(\mu)$ is said to be a Banach function space if it satisfies the following conditions:

(I) If $|f| \leq |g|$, with $f \in L_0(\mu)$ and $g \in X(\mu)$, then $f \in X(\mu)$ and $\|f\|_X \leq \|g\|_X$.

(II) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function χ_A of A belongs to $X(\mu)$.

For $1 \leq p \leq q \leq \infty$ we denote by $\mathbf{M}^{(\mathbf{p})}(X)$ and $\mathbf{M}_{(\mathbf{q})}(X)$ the p-convexity constant and the q-concavity constant of a Banach function space X with these properties (see e. g. [LT79]), and with $X(\mathbb{C})$ its natural complexification.

For all information on Banach operator ideals see e. g. [DF93], [DJT95], [Pie80] and [TJ89], and for the theory of tensor norms on tensor products of Banach spaces [DF93]. If E is a symmetric Banach sequence space, then we denote by $\mathcal{S}_E$ its associated unitary ideal, i. e. the space of all compact $T : \ell_2 \to \ell_2$ for which the sequence $(s_n(T))_n$ of singular numbers belongs to E, equipped with the norm $\|T\|_{\mathcal{S}_E} := \|(s_n(T))_n\|_E$.

We point out that, since we extensively use complex interpolation, the underlying field $\mathbb{K}$ is always $\mathbb{C}$—important exceptions are mentioned explicitly. However, many of our main results can be easily transferred to the real case; we leave this to the reader.

1 Complex interpolation and approximation

For an introduction to complex interpolation theory see [BL78] and [KPS82]. If we speak of a finite-dimensional interpolation couple $[E_0, E_1]$, we always assume that E_0 and E_1 have the same finite dimension. For $0 < \theta < 1$ and a complex interpolation couple we denote by $[E_0, E_1]_\theta$ the associated complex interpolation space, and if E_1 is a Hilbert space, then $[E_0, E_1]_\theta$ is called a θ-Hilbert space; this notion is due to Pisier. Note that one can always assume that $E_0 \cap E_1$ is dense in E_0 and E_1.

Complex interpolation of finite-dimensional Banach spaces is rather more interpolation of norms than interpolation of spaces which makes it sometimes easier to explain an idea, without struggling with topological questions. Therefore we try to prove most of our results in the finite-dimensional case first (with "uniform" control of certain constants), and then we

lift them to the infinite-dimensional case with the help of an "approximation" lemma which is proved in this first section.

For this purpose we introduce the notion of a "cofinal interpolation triple": If $[E_0, E_1]$ is an interpolation couple, $E \subset E_0 \cap E_1$ a subspace which is dense in E_0, E_1, and $\mathcal{B} \subset FIN(E)$ (where $FIN(E)$ denotes the collection of all finite-dimensional subspaces of E) is cofinal (i. e. for every $G \in FIN(E)$ there exists $M \in \mathcal{B}$ with $G \subset M$), then the triple $([E_0, E_1], E, \mathcal{B})$ is called a *cofinal interpolation triple.* For $M \in FIN(E)$ we denote by M_0 and M_1 the subspace M of E_0 and E_1 endowed with the induced norm, respectively. The following lemma seems to be folklore in Russian literature; a proof can be found in e. g. [Kou91, 4.1] or [Mic99, 1.3].

Lemma 1.1. *Let $([E_0, E_1], E, \mathcal{B})$ be a cofinal interpolation triple and $0 < \theta < 1$. Then for each $\varepsilon > 0$ and $G \in FIN(E)$ there exists $M \in \mathcal{B}$ such that $G \subset M$ and for all $x \in G$*

$$(1-\varepsilon) \cdot \|x\|_{[M_0,M_1]_\theta} \leq \|x\|_{[E_0,E_1]_\theta} \leq \|x\|_{[M_0,M_1]_\theta}.$$

Next we prove the approximation lemma announced above.

Lemma 1.2. *Let $([F_0, F_1], F, \mathcal{B})$ be a cofinal interpolation triple, $0 < \theta < 1$ and $(\mathcal{A}, A)$ a maximal Banach operator ideal with*

$$c_\theta := \sup_n \sup_{M \in \mathcal{B}} \|\mathcal{L}(\ell_1^n, [M_0, M_1]_\theta) \hookrightarrow \mathcal{A}(\ell_1^n, [M_0, M_1]_\theta)\| < \infty.$$

Then

$$\mathcal{L}(\ell_1, [F_0, F_1]_\theta) = \mathcal{A}(\ell_1, [F_0, F_1]_\theta).$$

Proof. Denote by ε the injective tensor norm, by α the finitely generated tensor norm associated to $(\mathcal{A}, A)$ and by $\overleftarrow{\alpha}$ its cofinite hull in the sense of [DF93, 17.3 and 12.4]. We prove that

$$\|\ell_\infty^n \otimes_\varepsilon (F, \|\cdot\|_{[F_0,F_1]_\theta}) \hookrightarrow \ell_\infty^n \otimes_{\overleftarrow{\alpha}} (F, \|\cdot\|_{[F_0,F_1]_\theta})\| \leq c_\theta. \tag{1.1}$$

Then by density (see [DF93, 13.4])

$$\|\ell_\infty^n \otimes_\varepsilon [F_0, F_1]_\theta \hookrightarrow \ell_\infty^n \otimes_{\overleftarrow{\alpha}} [F_0, F_1]_\theta\| \leq c_\theta,$$

hence the claim follows by the Embedding Theorem [DF93, 17.6] and local techniques [DF93, 23.1].

For $z \in \ell_\infty^n \otimes F$ choose by Lemma 1.1 a subspace $M \in \mathcal{B}$ such that $z \in \ell_\infty^n \otimes M$ and $\|(M, \|\cdot\|_{[F_0,F_1]_\theta}) \hookrightarrow [M_0, M_1]_\theta\| \leq 1 + \varepsilon$. Then by the mapping properties of $\overleftarrow{\alpha}$ and ε and the fact that the injective tensor norm respects subspaces we have

$$\|z\|_{\ell_\infty^n \otimes_{\overleftarrow{\alpha}} (F, \|\cdot\|_{[F_0,F_1]_\theta})} \leq \|z\|_{\ell_\infty^n \otimes_{\overleftarrow{\alpha}} [M_0,M_1]_\theta} \quad \text{and} \quad \|z\|_{\ell_\infty^n \otimes_\varepsilon [M_0,M_1]_\theta} \leq (1+\varepsilon) \cdot \|z\|_{\ell_\infty^n \otimes_\varepsilon (F, \|\cdot\|_{[F_0,F_1]_\theta})},$$

hence (1.1) follows from the assumption and the Embedding Theorem. □

In our applications the approximation lemma will be combined with what we call the "interpolation trick", due to Kwapień and based on complex interpolation of vector-valued ℓ_p's [BL78, 5.1.2]:

$$\mathcal{L}(\ell_1^n, [M_0, M_1]_\theta) = \ell_\infty^n([M_0, M_1]_\theta) = [\ell_\infty^n(M_0), \ell_\infty^n(M_1)]_\theta = [\mathcal{L}(\ell_1^n, M_0), \mathcal{L}(\ell_1^n, M_1)]_\theta$$

(isometrically).

2 Summing and mixing operators on ℓ_1

For all information on summing and mixing operators see e. g. [Pie80], [DJT95] and [DF93]. An operator $T \in \mathcal{L}(E, F)$ is called absolutely (r, p)-summing $(1 \leq p \leq r \leq \infty)$ if there is a constant $\rho \geq 0$ such that

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^r\Big)^{1/r} \leq \rho \cdot \sup\Big\{ \Big(\sum_{i=1}^{n} |\langle x', x_i\rangle|^p\Big)^{1/p} \,|\, x' \in B_{E'} \Big\}$$

for all finite sets of elements $x_1, \ldots, x_n \in E$ (with the obvious modifications for p or $r = \infty$). In this case, the infimum over all possible $\rho \geq 0$ is denoted by $\pi_{r,p}(T)$, and the maximal Banach operator ideal of all absolutely (r, p)-summing operators by $(\Pi_{r,p}, \pi_{r,p})$; the special case $r = p$ gives the ideal (Π_p, π_p) of all absolutely p-summing operators.

An operator $T \in \mathcal{L}(E, F)$ is called (s, p)-mixing $(1 \leq p \leq s \leq \infty)$ whenever its composition with an arbitrary operator $S \in \Pi_s(F, Y)$ is absolutely p-summing; with the norm

$$\mu_{s,p}(T) := \sup\{\pi_p(ST) \,|\, \pi_s(S) \leq 1\}$$

the class $\mathcal{M}_{s,p}$ of all (s, p)-mixing operators forms again a maximal Banach operator ideal. Obviously, $(\mathcal{M}_{p,p}, \mu_{p,p}) = (\mathcal{L}, \|\cdot\|)$ and $(\mathcal{M}_{\infty,p}, \mu_{\infty,p}) = (\Pi_p, \pi_p)$. Recall that due to [Mau74] (see also [DF93, 32.10–11]) summing and mixing operators are closely related:

$$(\mathcal{M}_{s,p}, \mu_{s,p}) \subset (\Pi_{r,p}, \pi_{r,p}) \qquad \text{for } 1/s + 1/r = 1/p, \tag{2.1}$$

and "conversely"

$$(\Pi_{r,p}, \pi_{r,p}) \subset (\mathcal{M}_{s_0,p}, \mu_{s_0,p}) \qquad \text{for } 1 \leq p \leq s_0 < s \leq \infty \text{ and } 1/s + 1/r = 1/p. \tag{2.2}$$

Moreover, it is known that the identity map id_X of a cotype 2 space X is $(2, 1)$-mixing and therefore every $(s, 2)$-mixing operator on a cotype 2 space is even $(s, 1)$-mixing (see again [Mau74] and [DF93, 32.2]). Finally, a quick investigation of [TJ70] shows that if $1 \leq q \leq 2 \leq r \leq \infty$ with $1/r = 1/q - 1/2$ are given, then $\Pi_{r,2}(X, \cdot) = \Pi_{q,1}(X, \cdot)$ for every Banach space X such that id_X is $(2, 1)$-mixing; by the above, this holds in particular for cotype 2 spaces, hence, most of our main results in this article can also be formulated in terms of $(q, 1)$-summing/mixing norms instead of $(q, 2)$-summing/mixing norms.

The following theorem is an extension of the results by Kwapień and [CD92]—it is an unpublished result from [ref92] and Lermer [Ler94] (however with proofs different from the one presented in this article).

Theorem 2.1. *Let F be a θ-Hilbert space, $0 < \theta < 1$. Then*

$$\mathcal{L}(\ell_1, F) = \mathcal{M}_{\frac{2}{1-\theta},2}(\ell_1, F) = \Pi_{\frac{2}{\theta},2}(\ell_1, F).$$

Since every 2-summing operator on ℓ_1 is 1-summing this implies

$$\mathcal{L}(\ell_1, F) = \mathcal{M}_{\frac{2}{1-\theta},1}(\ell_1, F) = \Pi_{\frac{2}{1+\theta},1}(\ell_1, F). \tag{2.3}$$

The proof is based on the following interpolation theorem for spaces of mixing operators in combination with Lemma 1.2.

Proposition 2.2. *Let $[E_0, E_1]$ and $[F_0, F_1]$ be finite-dimensional interpolation couples and $2 \le s_0, s_1 \le \infty$. Then for $0 \le \theta \le 1$ and $2 \le s_\theta \le \infty$ defined by $1/s_\theta = (1-\theta)/s_0 + \theta/s_1$*

$$\|[\mathcal{M}_{s_0,2}(E_0, F_0), \mathcal{M}_{s_1,2}(E_1, F_1)]_\theta \hookrightarrow \mathcal{M}_{s_\theta,2}([E_0, E_1]_\theta, [F_0, F_1]_\theta)\| \le d_\theta[E_0, E_1],$$

where $d_\theta[E_0, E_1] := \sup_m \|\mathcal{L}(\ell_2^m, [E_0, E_1]_\theta) \hookrightarrow [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta\|$.

Note that in the case $E_0 = E_1$ (isometrically), one trivially has $d_\theta[E_0, E_1] = 1$.

Proof. Consider for $\eta = 0, 1$ the mappings

$$\begin{array}{ccccccccc} \Phi_\eta^{n,m}: & \mathcal{M}_{s_\eta,2}(E_\eta, F_\eta) & \times & \ell_{s_\eta}^n(F'_\eta) & \times & \mathcal{L}(\ell_2^m, E_\eta) & \longrightarrow & \ell_2^m(\ell_{s_\eta}^n) \\ & T & \times & (y'_1, \dots, y'_n) & \times & S & \longmapsto & ((\langle y'_k, TSe_j\rangle)_k)_j \end{array},$$

where (e_j) denotes the canonical basis in $\mathbb{C}^m$. By the discrete characterization of the mixing norm (see [Mau74] or [DF93, 32.4]) $\mu_{s_\eta,2}(T : E_\eta \to F_\eta)$ is the infimum over all $c \ge 0$ such that for all n, m, all $y'_1, \dots, y'_n \in F'_\eta$ and all $x_1, \dots, x_m \in E_\eta$

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n |\langle y'_k, Tx_j\rangle|^{s_\eta} \right)^{2/s_\eta} \right)^{1/2} \le c \cdot \left(\sum_{k=1}^n \|y'_k\|_{F'_\eta}^{s_\eta} \right)^{1/s_\eta} \cdot \sup_{x' \in B_{E'_\eta}} \left(\sum_{j=1}^m |\langle x', x_j\rangle|^2 \right)^{1/2}.$$

Since for each $S = \sum_{j=1}^m e_j \otimes x_j \in \mathcal{L}(\ell_2^m, E_\eta)$

$$\|S\| = \sup_{x' \in B_{E'_\eta}} \left(\sum_{j=1}^m |\langle x', x_j\rangle|^2 \right)^{1/2},$$

it clearly follows that $\|\Phi_\eta^{n,m}\| \le 1$. Then for the interpolated mapping

$$\begin{aligned} [\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta : [\mathcal{M}_{s_0,2}(E_0, F_0), \mathcal{M}_{s_1,2}(E_1, F_1)]_\theta \times [\ell_{s_0}^n(F'_0), \ell_{s_1}^n(F'_1)]_\theta \\ \times [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta \to [\ell_2^m(\ell_{s_0}^n), \ell_2^m(\ell_{s_1}^n)]_\theta, \end{aligned}$$

by multilinear interpolation (see e. g. [BL78, 4.4.1]) we also have $\|[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta\| \le 1$. It follows that for each $T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta$, each $S \in \mathcal{L}(\ell_2^m, [E_0, E_1]_\theta)$ and $y'_1, \dots, y'_n \in [F_0, F_1]'_\theta$

$$\begin{aligned} &\left(\sum_{j=1}^m \left(\sum_{k=1}^n |\langle y'_k, TSe_j\rangle|^{s_\theta} \right)^{2/s_\theta} \right)^{1/2} \\ &\quad \le \|T\|_{[\mathcal{M}_{s_0,2}(E_0,F_0), \mathcal{M}_{s_1,2}(E_1,F_1)]_\theta} \cdot \|S\|_{[\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta} \cdot \|(y_k)_k\|_{[\ell_{s_0}^n(F'_0), \ell_{s_1}^n(F'_1)]_\theta} \\ &\quad \le d_\theta[E_0, E_1] \cdot \|T\|_{[\mathcal{M}_{s_0,2}(E_0,F_0), \mathcal{M}_{s_1,2}(E_1,F_1)]_\theta} \cdot \|S\|_{\mathcal{L}(\ell_2^m, [E_0,E_1]_\theta)} \cdot \|(y_k)_k\|_{\ell_{s_\theta}^n([F_0,F_1]'_\theta)}, \end{aligned}$$

hence

$$\|T\|_{\mathcal{M}_{s_\theta,2}([E_0,E_1]_\theta, [F_0,F_1]_\theta)} \le d_\theta[E_0, E_1] \cdot \|T\|_{[\mathcal{M}_{s_0,2}(E_0,F_0), \mathcal{M}_{s_1,2}(E_1,F_1)]_\theta}.$$

□

Proof of Theorem 2.1: Let $F = [F_0, F_1]_\theta$ where F_1 is a Hilbert space and $F_0 \cap F_1$ is dense in F_0 and F_1 (this implies that $([F_0, F_1], F_0 \cap F_1, FIN(F_0 \cap F_1))$ is a cofinal interpolation triple), and let $M \in FIN(F_0 \cap F_1)$. Then by the Little Grothendieck Theorem (see e. g. [DF93, 11.11])

$$\|\mathcal{L}(\ell_1^n, M_1) \hookrightarrow \mathcal{M}_{\infty,2}(\ell_1^n, M_1) = \Pi_2(\ell_1^n, M_1)\| \leq K_{LG},$$

and trivially $\|\mathcal{L}(\ell_1^n, M_0) \hookrightarrow \mathcal{M}_{2,2}(\ell_1^n, M_0) = \mathcal{L}(\ell_1^n, M_0)\| \leq 1$, hence, by the usual interpolation theorem together with Proposition 2.2 and Kwapień's interpolation trick, we have

$$\|\mathcal{L}(\ell_1^n, [M_0, M_1]_\theta) = [\mathcal{L}(\ell_1^n, M_0), \mathcal{L}(\ell_1^n, M_1)]_\theta \hookrightarrow \mathcal{M}_{2/(1-\theta),2}(\ell_1^n, [M_0, M_1]_\theta)\| \leq K_{LG}^\theta$$

$(d_\theta[\ell_1^n, \ell_1^n] = 1)$. The claim now follows by Lemma 1.2. □

We illustrate the preceding theorem by several examples and a corollary (see the remarks after the proof of the corollary for credits):

Example 2.3. For $1 < p < \infty$, $p \neq 2$, and $\theta = 1 - |1 - 2/p|$ it is known that ℓ_p and $\mathcal{S}_p$ are θ-Hilbert spaces (see e. g. [BL78, 5.1.1] and [PT68, Satz 8] together with the complex reiteration theorem [BL78, 4.6.1]), hence we obtain

$$\mathcal{L}(\ell_1, \ell_p) = \mathcal{M}_{2/|1-2/p|,2}(\ell_1, \ell_p) = \Pi_{2/(1-|1-2/p|),2}(\ell_1, \ell_p), \tag{2.4}$$

and as a sort of non-commutative analogue

$$\mathcal{L}(\ell_1, \mathcal{S}_p) = \mathcal{M}_{2/|1-2/p|,2}(\ell_1, \mathcal{S}_p) = \Pi_{2/(1-|1-2/p|),2}(\ell_1, \mathcal{S}_p). \tag{2.5}$$

Example 2.4. More generally, Pisier [Pi79] showed that for $1 < p < 2$ every p-convex and p'-concave Banach lattice X is a $2/p'$-Hilbert space, therefore

$$\mathcal{L}(\ell_1, X) = \mathcal{M}_{2p/(2-p),2}(\ell_1, X) = \Pi_{p',2}(\ell_1, X), \tag{2.6}$$

and if in addition X is a symmetric Banach sequence space, then by e. g. [Ara78] it is known that $\mathcal{S}_X$ is also a $2/p'$-Hilbert space, which gives

$$\mathcal{L}(\ell_1, \mathcal{S}_X) = \mathcal{M}_{2p/(2-p),2}(\ell_1, \mathcal{S}_X) = \Pi_{p',2}(\ell_1, \mathcal{S}_X). \tag{2.7}$$

Example 2.5. Let $0 < \theta' < \theta < 1$ and $2/(2-\theta) \leq q \leq 2/\theta$. Then by [Mat89, Theorem C] every (θ, q)-Hilbert space F (this means that $F = [F_0, F_1]_{\theta,q}$ for some interpolation couple $[F_0, F_1]$ and F_1 a Hilbert space) is isomorphic to a subspace of a θ'-Hilbert space, hence for such a Banach space F Theorem 2.1 holds for θ' instead of θ. The most prominent examples for (θ, q)-Hilbert spaces are Lorentz spaces $\ell_{p,q}$ and their associated unitary ideals $\mathcal{S}_{p,q}$—but note that the results for these spaces are also included in Example 2.4.

Trace duality allows interesting reformulations of Theorem 2.1.

Corollary 2.6. *Let F be a θ-Hilbert space,* $0 < \theta < 1$.

(a) $\mathcal{L}(\ell_\infty, F) = \mathcal{M}_{\frac{2}{1-\theta},2}(\ell_\infty, F) = \Pi_{\frac{2}{\theta},2}(\ell_\infty, F)$.

(b) Every $\frac{2}{1-\theta}$-summing operator on F factorizes through a Hilbert space.

Proof. (b) follows from (2.3) by trace duality: By local techniques (see again [DF93, 23.1]) statement (2.3) in terms of quotient ideals (see e. g. [DF93, 25.6]) reads as follows:

$$\Pi_{\frac{2}{1-\theta}}(F,\cdot) \subset (\Pi_1 \circ \Gamma_1^{-1})(F,\cdot),$$

where Γ_p for $1 \le p \le \infty$ stands for the Banach operator ideal of all $T : F \to Y$ such that $F \xrightarrow{T} Y \hookrightarrow Y''$ factorizes through some $L_p(\mu)$. Hence the abstract quotient formula from [DF93, 25.7] together with the trace formula $\Pi_1^* = \mathcal{I}_\infty = \Gamma_\infty$ (see e. g. [DJT95, 6.16]) and the fact that the adjoint Γ_2^* of Γ_2 is contained in $\Gamma_1 \circ \Gamma_\infty$ (a result of Kwapień, see e. g. [DJT95, 7.12]) imply the conclusion:

$$\Pi_{\frac{2}{1-\theta}}(F,\cdot) \subset (\Gamma_1 \circ \Gamma_\infty)^* \subset \Gamma_2.$$

Finally, (a) is an immediate consequence of (b): Take $T \in \mathcal{L}(\ell_\infty, F)$ and some $S \in \Pi_{\frac{2}{1-\theta}}(F,Y) \subset \Gamma_2(F,Y)$ (by (b)), Y some Banach space. Then ST by the little Grothendieck theorem is 2-summing. □

For $\theta = 1$ and $F = \ell_2$ the statements (2.4) and Corollary 2.6 (a) are the "Little Grothendieck Theorems". The special cases $\mathcal{L}(\ell_1, \ell_p) = \Pi_{r,2}(\ell_1, \ell_p)$, $1/r = 1/2 - |1/2 - 1/p|$, and $\mathcal{L}(\ell_\infty, \ell_p) = \Pi_{p,2}(\ell_\infty, \ell_p)$, $2 \le p \le \infty$, are due to Kwapień [Kwa68] and Lindenstrauss–Pełczyński [LP68], respectively, whereas Example 2.4 for summing operators is contained in [Pi79]. Example 2.5 is due to Lermer [Ler94], and Corollary 2.6 in the present form seems to be new (in the case $F = \ell_p$ see [DF93, Ex. 34.12] and [DJT95, p. 168]).

Corollary 2.7. *Let $2 \le r, s \le \infty$ and $1/r = 1/2 - 1/s$.*

(a) $\mathcal{S}_r = \Pi_{r,2}(\ell_2) = \mathcal{M}_{s,2}(\ell_2)$ (isometrically).

(b) $\Pi_{r,2}(E, \ell_2) = \mathcal{M}_{s,2}(E, \ell_2)$ for every Banach space E.

The first equality in (a) is due to Mitiagin and was first published in [Kwa68] (see e. g. [Kön86, 1.d.12] or [DJT95, 10.3] for an elementary proof). The second equality in (a) was proved in [CD92]—here is an alternative proof by interpolation and the first equality: By Proposition 2.2 for θ defined by $1/r = (1-\theta)/2$ and the first equality the embeddings in

$$\mathcal{S}_r^n = [\mathcal{S}_2^n, \mathcal{S}_\infty^n]_\theta = [\mathcal{M}_{\infty,2}(\ell_2^n), \mathcal{M}_{2,2}(\ell_2^n)]_\theta \hookrightarrow \mathcal{M}_{s,2}(\ell_2^n) \hookrightarrow \Pi_{r,2}(\ell_2^n) = \mathcal{S}_r^n$$

all have norm ≤ 1, and by localization this gives the claim. Now it is easy to prove (b), which is a kind of extension of (a): Since $\mathcal{M}_{s,2} = \mathcal{I}_{s'} \circ \Pi_2^{-1}$ (see e. g. [DF93, 32.1]; $\mathcal{I}_{s'}$ denotes the ideal of s'-integral operators), it suffices to show that $TS \in \mathcal{I}_{s'}$ whenever $T \in \Pi_{r,2}(E, \ell_2)$ and $S \in \Pi_2(X, E)$. But by the Grothendieck–Pietsch factorization theorem (see e. g. [DF93, 11.3]) we know that $S = UV$ where $V \in \Pi_2(X, H)$, $U \in \mathcal{L}(H, E)$ and H a Hilbert space. Then by (a) and local techniques $TU \in \Pi_{r,2}(H, \ell_2) = \mathcal{M}_{s,2}(H, \ell_2)$, which gives $TS \in \mathcal{I}_{s'}$.

3 Rademacher and Gaussian p-summing operators on ℓ_1

For $1 \le p \le 2$ Kwapień's result reads as follows:

$$\mathcal{L}(\ell_1, \ell_p) = \Pi_{r,1}(\ell_1, \ell_p) = \Pi_{2,p}(\ell_1, \ell_p), \quad 1/r = 3/2 - 1/p, \tag{3.1}$$

where the last equality follows by the usual inclusion formulas for summing operators (see e. g. [DJT95, 10.4]).

We now extend (3.1) within the framework of Rademacher and Gaussian p-summing operators—we will "replace the 2 in the last equality by Rademacher or Gauss functions". For $1 \le p \le 2$ an operator $T : X \to Y$ between Banach spaces X and Y is said to belong to the class of *Rademacher p-summing operators* ($\Pi_{\mathcal{R},p}$ for short) if there exists a constant $c > 0$ such that for all finite sequences $x_1, \dots, x_n$ in X

$$\left(\int_0^1 \| \sum_{i=1}^n r_i(t) \cdot T x_i \|^2 dt \right)^{1/2} \le c \cdot \sup_{x' \in B_{X'}} \left(\sum_{i=1}^n |\langle x', x_i \rangle|^p \right)^{1/p}, \tag{3.2}$$

where $(r_i)_i$ denotes as usual the sequence of Rademacher functions; these are all operators which transform weakly p-summable sequences into almost unconditionally summable sequences (to see this use e. g. [DJT95, 12.3]). We write $\pi_{\mathcal{R},p}(T)$ for the smallest constant c satisfying (3.2) and obtain in this way the injective and maximal Banach operator ideal $(\Pi_{\mathcal{R},p}, \pi_{\mathcal{R},p})$. If in (3.2) the Rademacher functions are substituted by a sequence of independent Gaussian variables $(g_i)_i$, then we denote the resulting Banach operator ideal of all *Gaussian p-summing operators* by $(\Pi_{\gamma,p}, \pi_{\gamma,p})$; for $p = 2$ it was originally introduced by [LP74]. Note that $\Pi_{\mathcal{R},2}$ and $\Pi_{\gamma,2}$ coincide (see e. g. [DJT95, 12.12]), and if Y has non-trivial cotype, then $\Pi_{\gamma,p}(X, Y) = \Pi_{\mathcal{R},p}(X, Y)$ for any Banach space X (use [DJT95, 12.11 and 12.27]). Finally, $\mathcal{L} = \Pi_{\mathcal{R},1}$ (apply the Kahane inequality [DJT95, 11.1]).

Let $0 < \theta < 1$. A Banach space F is called a *θ-type 2 space* if there exists an interpolation couple $[F_0, F_1]$ such that F_1 has type 2 and $F = [F_0, F_1]_\theta$. As in the case of θ-Hilbert spaces one can always assume that $F_0 \cap F_1$ is dense in F_0 and F_1.

Our extension of Kwapień's result (3.1) within the framework of Rademacher p-summing operators now reads as follows:

Theorem 3.1. *For $1 < p < 2$ and $\theta := 2/p'$ let F be a θ-type 2 space. Then*

$$\mathcal{L}(\ell_1, F) = \Pi_{\mathcal{R},p}(\ell_1, F).$$

Note that by interpolation a θ-type 2 space F, and therefore by K-convexity and duality also its dual space F', have non-trivial type, which implies that F has non-trivial cotype. Hence, in the above theorem one may substitute $\Pi_{\mathcal{R},p}(\ell_1, F)$ by $\Pi_{\gamma,p}(\ell_1, F)$.

Before we give the proof let us again illustrate our result by a first example.

Example 3.2. Since every θ-Hilbert space is a θ-type 2 space, ℓ_p and $\mathcal{S}_p$ for $1 < p < 2$ are $2/p'$-type 2 spaces (see Example 2.3). Hence, by Theorem 3.1

$$\mathcal{L}(\ell_1, \ell_p) = \Pi_{\mathcal{R},p}(\ell_1, \ell_p) \qquad \text{and} \qquad \mathcal{L}(\ell_1, \mathcal{S}_p) = \Pi_{\mathcal{R},p}(\ell_1, \mathcal{S}_p).$$

This gives an alternative approach to Kwapień's result in the case $1 < p < 2$: ℓ_p and $\mathcal{S}_p$ both have cotype 2 (the latter result is due to [TJ74]), hence, by the definition of the Rademacher p-summing operators, for $V_p = \ell_p$ or $\mathcal{S}_p$

$$\mathcal{L}(\ell_1, V_p) = \Pi_{\mathcal{R},p}(\ell_1, V_p) \subset \Pi_{2,p}(\ell_1, V_p) = \Pi_{r,1}(\ell_1, V_p),$$

where $1/r = 3/2 - 1/p$ (see the preliminaries in Section 2 for the last equality).

In our proof of Theorem 3.1 we interpolate between the well-known fact that a Banach space X has type 2 if and only if $\mathcal{L}(\ell_1, X) = \Pi_{\mathcal{R},2}(\ell_1, X)$ (see e. g. [DJT95, 12.10]) and the simple observation $\mathcal{L} = \Pi_{\mathcal{R},1}$ from above. For this we need an analogue to Proposition 2.2. We define (as an extension of the definition of $d_\theta[\cdot,\cdot]$) for a finite-dimensional interpolation couple $[E_0, E_1]$, $0 < \theta < 1$, and $1 \le p_0, p_1, p_\theta \le 2$ such that $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$

$$d_\theta[\ell_{p_0}, \ell_{p_1}; E_0, E_1] := \sup_m \|\mathcal{L}(\ell^m_{p'_\theta}, [E_0, E_1]_\theta) \hookrightarrow [\mathcal{L}(\ell^m_{p'_0}, E_0), \mathcal{L}(\ell^m_{p'_1}, E_1)]_\theta\|.$$

By [Kou91] (see also [DM99, Proposition 8]) this quantity is always bounded, and the estimate $d_\theta[\ell_1, \ell_2; \ell^n_u, \ell^n_u] \le 16$ for all $1 \le u \le 2$ will be crucial for our applications.

Proposition 3.3. *Let $[E_0, E_1]$ and $[F_0, F_1]$ be finite-dimensional interpolation couples. Then for $1 \le p_0, p_1 \le 2$ and $0 < \theta < 1$*

$$\|[\Pi_{\mathcal{R},p_0}(E_0, F_0), \Pi_{\mathcal{R},p_1}(E_1, F_1)]_\theta \hookrightarrow \Pi_{\mathcal{R},p_\theta}([E_0, E_1]_\theta, [F_0, F_1]_\theta)\| \le d_\theta[\ell_{p_0}, \ell_{p_1}; E_0, E_1],$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

Proof. Throughout the proof we denote $L_2 := L_2[0,1]$. Consider for $\eta = 0, 1$ the bilinear mapping

$$\begin{array}{rcccl} \Phi^m_\eta : & \Pi_{\mathcal{R},p_\eta}(E_\eta, F_\eta) & \times & \mathcal{L}(\ell^m_{p'_\eta}, E_\eta) & \to & L_2(F_\eta) \\ & T & \times & S & \mapsto & \sum_{i=1}^m r_i \cdot TSe_i. \end{array}$$

By definition $\|\Phi^m_\eta\| \le 1$, hence for the interpolated mapping

$$\begin{aligned} [\Phi^m_0, \Phi^m_1]_\theta : [\Pi_{\mathcal{R},p_0}(E_0, F_0), \Pi_{\mathcal{R},p_1}(E_1, F_1)]_\theta \times [\mathcal{L}(\ell^m_{p'_0}, E_0), \mathcal{L}(\ell^m_{p'_1}, E_1)]_\theta \\ \to [L_2(F_0), L_2(F_1)]_\theta = L_2([F_0, F_1]_\theta) \end{aligned}$$

we obtain $\|[\Phi^m_0, \Phi^m_1]_\theta\| \le 1$. It follows that for each $T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta$ and each $S \in \mathcal{L}(\ell^m_{p'_\theta}, [E_0, E_1]_\theta)$

$$\begin{aligned} &\left\| \sum_{i=1}^m r_i \cdot TSe_i \right\|_{L_2([F_0,F_1]_\theta)} \\ &\qquad \le \|T\|_{[\Pi_{\mathcal{R},p_0}(E_0,F_0), \Pi_{\mathcal{R},p_1}(E_1,F_1)]_\theta} \cdot \|S\|_{[\mathcal{L}(\ell^m_{p'_0},E_0), \mathcal{L}(\ell^m_{p'_1},E_1)]_\theta} \\ &\qquad \le d_\theta[\ell_{p_0}, \ell_{p_1}; E_0, E_1] \cdot \|T\|_{[\Pi_{\mathcal{R},p_0}(E_0,F_0), \Pi_{\mathcal{R},p_1}(E_1,F_1)]_\theta} \cdot \|S\|_{\mathcal{L}(\ell^m_{p'_\theta},[E_0,E_1]_\theta)}, \end{aligned}$$

hence

$$\|T\|_{\Pi_{\mathcal{R},p_\theta}([E_0,E_1]_\theta,[F_0,F_1]_\theta)} \le d_\theta[\ell_{p_0}, \ell_{p_1}; E_0, E_1] \cdot \|T\|_{[\Pi_{\mathcal{R},p_0}(E_0,F_0), \Pi_{\mathcal{R},p_1}(E_1,F_1)]_\theta}.$$

□

Proof of Theorem 3.1: Let $F = [F_0, F_1]_\theta$ where F_1 has type 2 and $F_0 \cap F_1$ is dense in F_0 and F_1 (this again implies that $([F_0, F_1], F_0 \cap F_1, FIN(F_0 \cap F_1))$ is a cofinal interpolation triple), and let $M \in FIN(F_0 \cap F_1)$. An easy application of the Kahane inequality yields

$$\|\mathcal{L}(\ell^n_1, M_0) \hookrightarrow \Pi_{\mathcal{R},1}(\ell^n_1, M_0)\| \le \sqrt{2},$$

and by [DJT95, p. 245]

$$\|\mathcal{L}(\ell_1^n, M_1) \hookrightarrow \Pi_{\mathcal{R},2}(\ell_1^n, M_1)\| \leq K_G \cdot \mathbf{T_2}(F_1),$$

where K_G denotes the Grothendieck constant. Hence, by the usual interpolation theorem together with Kwapień's interpolation trick and Proposition 3.3, we obtain (recall that $d_\theta[\ell_1, \ell_2; \ell_1^n, \ell_1^n] \leq 16$)

$$\|\mathcal{L}(\ell_1^n, [M_0, M_1]_\theta) \hookrightarrow \Pi_{\mathcal{R},p}(\ell_1^n, [M_0, M_1]_\theta)\| \leq 16 \cdot 2^{(1-\theta)/2} \cdot (K_G \cdot \mathbf{T_2}(F_1))^\theta.$$

The claim now follows by Lemma 1.2. □

For our second example the following extrapolation theorem is crucial:

Theorem 3.4. *For $1 < p < 2$ let X be a p-convex Banach function space with finite concavity. Then $X(\mathbb{C})$ is a θ-type 2 space, $\theta := 2/p'$.*

For the proof of this theorem we need some further tools. Let $X_0(\mu), X_1(\mu)$ be Banach function spaces and $0 < \theta < 1$. The Banach function space $X_0^{1-\theta} X_1^\theta$ is defined to be the set of functions $f \in L_0(\mu)$ for which there exist $g \in X_0$ and $h \in X_1$ such that $|f| = |g|^{1-\theta} \cdot |h|^\theta$, together with the norm

$$\|f\|_{X_0^{1-\theta} X_1^\theta} := \inf\{\|g\|_{X_0}^{1-\theta} \cdot \|h\|_{X_1}^\theta \,|\, |f| = |g|^{1-\theta} \cdot |h|^\theta, g \in X_0, h \in X_1\}.$$

For $0 < r < \infty$ and a Banach function space $X(\mu)$ with $\mathbf{M}^{(\max(\mathbf{1},\mathbf{r}))}(X) = 1$ we define the Banach function space $(X^r, \|\cdot\|_{X^r})$ by

$$X^r := \{f \in L_0(\mu) \,|\, |f|^{1/r} \in X\} \qquad \text{and} \qquad \|f\|_{X^r} := \||f|^{1/r}\|_X^r, \quad f \in X^r.$$

An easy calculation (see e. g. [Def99, Lemma 2]) shows that X^r is (s/r)-convex and (t/r)-concave whenever $\max(1, r) \leq t, s < \infty$ and X is s-convex and t-concave.

The following crucial interpolation formula for Banach function spaces is due to Calderón [Cal64, 13.6]: Let $X_0(\mu), X_1(\mu)$ be Banach function spaces such that at least one is σ-order continuous. Then for each $0 < \theta < 1$ isometrically

$$[X_0(\mathbb{C}), X_1(\mathbb{C})]_\theta = (X_0^{1-\theta} X_1^\theta)(\mathbb{C}). \tag{3.3}$$

Now we are able to state the following interpolation/extrapolation lemma.

Lemma 3.5. *Let $X_0(\mu)$ and $X_1(\mu)$ be Banach function spaces, and for $0 < r < 1 < p < \infty$ assume that X_0 is p-convex with $\mathbf{M}^{(\mathbf{p})}(X_0) = 1$. Then for $0 < \theta < 1$ such that $p(1-\theta) + r\theta = 1$*

$$(X_0^p)^{1-\theta} (X_1^r)^\theta = X_0^{p(1-\theta)} X_1^{r\theta}.$$

Moreover, if X is a p-convex Banach function space with $\mathbf{M}^{(\mathbf{p})}(X) = 1$, then for $\theta := 2/p'$

$$X = (X^p)^{1-\theta} (X^{p/2})^\theta.$$

Proof. Let $V := (X_0^p)^{1-\theta}(X_1^r)^\theta$ and $W := X_0^{p(1-\theta)}X_1^{r\theta}$. Then, if $f \in V$ and $|f| = |g|^{1-\theta} \cdot |h|^\theta$ with $g \in X_0^p$ and $h \in X_1^r$, then $|f| = (|g|^{1/p})^{p(1-\theta)} \cdot (|h|^{1/r})^{r\theta} \in W$, and

$$\|f\|_W = \|(|g|^{1/p})^{p(1-\theta)} \cdot (|h|^{1/r})^{r\theta}\|_W \le \||g|^{1/p}\|_{X_0}^{p(1-\theta)} \cdot \||h|^{1/r}\|_{X_1}^{r\theta} = \|g\|_{X_0^p}^{1-\theta} \cdot \|h\|_{X_1^r}^{\theta},$$

hence $\|f\|_W \le \|f\|_V$. Conversely, let $f \in W$ and $|f| = |g|^{p(1-\theta)} \cdot |h|^{r\theta}$ with $g \in X_0$ and $h \in X_1$. Then $|f| = (|g|^p)^{1-\theta} \cdot (|h|^r)^\theta \in V$, and

$$\|f\|_V = \|(|g|^p)^{1-\theta} \cdot (|h|^r)^\theta\|_V \le \||g|^p\|_{X_0^p}^{1-\theta} \cdot \||h|^r\|_{X_1^r}^{\theta} = \|g\|_{X_0}^{p(1-\theta)} \cdot \|h\|_{X_1}^{r\theta},$$

hence $\|f\|_V \le \|f\|_W$. For the rest observe that $X = X^{1-\eta}X^\eta$ holds isometrically with equal norms for each $0 < \eta < 1$; this follows from the abstract Hölder inequality (see e. g. [LT79, 1.d.2]): Let $f \in X^{1-\eta}X^\eta$ and $|f| = |g|^{1-\eta} \cdot |h|^\eta$ with $g, h \in X$. Then $|f| = |g|^{1-\eta} \cdot |h|^\eta \in X$, and

$$\|f\|_X = \||g|^{1-\eta} \cdot |h|^\eta\|_X \le \|g\|_X^{1-\eta} \cdot \|h\|_X^\eta,$$

hence $\|f\|_X \le \|f\|_{X^{1-\eta}X^\eta}$. Conversely, we have $|f| = |f|^{1-\eta} \cdot |f|^\eta \in X^{1-\eta}X^\eta$, and

$$\|f\|_{X^{1-\eta}X^\eta} \le \|f\|_X^{1-\eta} \cdot \|f\|_X^\eta = \|f\|_X.$$

Clearly $\theta := 2/p'$ satisfies $p(1-\theta) + p/2 \cdot \theta = 1$. Altogether we obtain that

$$(X^p)^{1-\theta}(X^{p/2})^\theta = X^{p(1-\theta)}X^{p\theta/2} = X.$$

$\square$

Proof of Theorem 3.4: Without loss of generality we may assume that $\mathbf{M}^{(\mathbf{p})}(X) = 1$ (see e. g. [LT79, 1.d.8]). By assumption there exists $p < q < \infty$ such that X is q-concave. Then $X^{p/2}$ is 2-convex and $(2q/p)$-concave which implies that $X^{p/2}$ is σ-order continuous (see [LT79, 1.a.5 and 1.a.7]) and that $X^{p/2}(\mathbb{C})$ has type 2 (see [LT79, 1.f.17]). Hence, Lemma 3.5 and (3.3) imply that $X(\mathbb{C}) = [X^p(\mathbb{C}), X^{p/2}(\mathbb{C})]_\theta$ is a θ-type 2 space, with $\theta := 2/p'$. $\square$

Example 3.6. For $1 < p < 2$ let X be a p-convex Banach function space with finite concavity. Then by Theorem 3.4 we know that $X(\mathbb{C})$ is a θ-type 2 space, $\theta := 2/p'$, and consequently (by complexification)

$$\mathcal{L}(\ell_1, X) = \Pi_{\mathcal{R},p}(\ell_1, X).$$

If in addition X is a symmetric Banach sequence space, then $\mathcal{S}_X = [\mathcal{S}_{X^p}, \mathcal{S}_{X^{p/2}}]_\theta$ (by [Ara78]) is also a θ-type 2 space ($\mathcal{S}_{X^{p/2}}$ has type 2 by [GTJ83, p. 190]), hence

$$\mathcal{L}(\ell_1, \mathcal{S}_X) = \Pi_{\mathcal{R},p}(\ell_1, \mathcal{S}_X).$$

Recall that a Banach space X has type 2 if and only if $\mathcal{L}(\ell_1, X) = \Pi_{\mathcal{R},2}(\ell_1, X)$; Example 3.6 now reveals that the ideal $\Pi_{\mathcal{R},p}$ might play a similar role for the notion of type p $(1 < p < 2)$, at least for Banach function spaces or unitary ideals. If we define as usual

$$p(X) := \sup\{1 \le p \le 2 \,|\, X \text{ has type } p\},$$

and in addition

$$p_{\mathcal{R}}(X) := \sup\{1 \le p \le 2 \,|\, \mathcal{L}(\ell_1, X) = \Pi_{\mathcal{R},p}(\ell_1, X)\},$$

we obtain the following:

Corollary 3.7. *Let X be a Banach function space or a unitary ideal. Then $p(X) = p_{\mathcal{R}}(X)$.*

Proof. We start with the case where X is a Banach function space. Let $p(X) > 1$. Then by [LT79, 1.f.9] and [LT79, 1.f.13] X is p-convex for all $1 \leq p < p(X)$ and q-concave for some $q < \infty$, hence by Example 3.6 the equality $\mathcal{L}(\ell_1, X) = \Pi_{\mathcal{R},p}(\ell_1, X)$ holds for all $1 \leq p < p(X)$, and consequently $p_{\mathcal{R}}(X) \geq p(X)$. If $X = \mathcal{S}_E$ with a symmetric Banach sequence space E and $p(\mathcal{S}_E) > 1$, then E has type p for all $1 < p < p(\mathcal{S}_E)$. Consequently, as above, E is p-convex for all $1 \leq p < p(\mathcal{S}_E)$ and q-concave for some $q < \infty$. Hence, by Example 3.6, the equality $\mathcal{L}(\ell_1, \mathcal{S}_E) = \Pi_{\mathcal{R},p}(\ell_1, \mathcal{S}_E)$ holds for all $1 \leq p < p(\mathcal{S}_E)$, and therefore $p_{\mathcal{R}}(\mathcal{S}_E) \geq p(\mathcal{S}_E)$.

Conversely, if $p_{\mathcal{R}}(X) > 1$, then by similar arguments as in [DJT95, p. 237] the Banach space X is of type p for all $1 \leq p < p_{\mathcal{R}}(X)$, hence $p(X) \geq p_{\mathcal{R}}(X)$. □

The preceding result leads to the following natural questions: Does $p(X) = p_{\mathcal{R}}(X)$ hold for every Banach space X? Furthermore: Is for $1 < p < 2$ a Banach space X of type p if and only if $\mathcal{L}(\ell_1, X) = \Pi_{\mathcal{R},p}(\ell_1, X)$?

4 Eigenvalue distributions of nuclear operators on θ-type 2 spaces

König in [Kön86, p. 110] shows that if F is a Banach space for which there exists some $2 < s < \infty$ satisfying

$$\mathcal{L}(\ell_1, F) = \Pi_{s,2}(\ell_1, F), \tag{4.1}$$

then each nuclear operator T on F has r-th power summable eigenvalues for some $1 < r < 2$ (recall that such T has always eigenvalues which are at least in ℓ_2, [Kön86, 2.b.13]).

Moreover, he conjectures that for each Banach space with non-trivial type there exists an s as in (4.1). The following theorem gives an affirmative answer for the subclass of all θ-type 2 spaces, $0 < \theta < 1$.

Theorem 4.1. *For $0 < \theta < 1$ let F be a θ-type 2 space. Then there exist $2 < s, t < \infty$ (with $1/s + 1/t = 1/2$) such that*

$$\mathcal{L}(\ell_1, F) = \mathcal{M}_{s,2}(\ell_1, F) = \Pi_{t,2}(\ell_1, F).$$

In particular, there exists $1 < r < 2$ such that every nuclear operator on F has r-th power summable eigenvalues.

Proof. Let $F = [F_0, F_1]_\theta$ where F_1 has type 2 and $F_0 \cap F_1$ is dense in F_0 and F_1, and let $M \in FIN(F_0 \cap F_1)$. As in the proof of Theorem 3.1 we know that

$$\|\mathcal{L}(\ell_1^n, M_1) \hookrightarrow \Pi_{\mathcal{R},2}(\ell_1, M_1)\| \leq K_G \cdot \mathbf{T_2}(F_1),$$

and since F_1 has type 2 it has cotype q for some $2 \leq q < \infty$, hence

$$\|\Pi_{\mathcal{R},2}(\ell_1^n, M_1) \hookrightarrow \Pi_{q,2}(\ell_1^n, M_1)\| \leq \mathbf{C_q}(F_1).$$

For each $2 < p < \infty$ such that $1/p + 1/q > 1/2$ there exists (by the "converse" inclusion formula for summing/mixing operators (2.2)) a universal constant $C_{p,q}$ such that

$$\|\Pi_{q,2}(\ell_1^n, M_1) \hookrightarrow \mathcal{M}_{p,2}(\ell_1^n, M_1)\| \le C_{p,q},$$

and altogether we obtain

$$\|\mathcal{L}(\ell_1^n, M_1) \hookrightarrow \mathcal{M}_{p,2}(\ell_1^n, M_1)\| \le C_{p,q} \cdot K_G \cdot \mathbf{T_2}(F_1) \cdot \mathbf{C_q}(F_1).$$

Together with

$$\|\mathcal{L}(\ell_1^n, M_0) \hookrightarrow \mathcal{M}_{2,2}(\ell_1^n, M_0)\| = 1,$$

the usual interpolation theorem, Kwapień's interpolation trick and Proposition 2.2 this gives for $2 < s < \infty$ such that $1/s = (1-\theta)/2 + \theta/p$

$$\|\mathcal{L}(\ell_1^n, [M_0, M_1]_\theta) \hookrightarrow \mathcal{M}_{s,2}(\ell_1^n, [M_0, M_1]_\theta)\| \le (C_{p,q} \cdot K_G \cdot \mathbf{T_2}(F_1) \cdot \mathbf{C_q}(F_1))^\theta.$$

The claim now follows by Lemma 1.2 and the inclusion formula for summing/mixing operators (2.1). □

Note that Banach function spaces with non-trivial type by Theorem 3.4 are θ-type 2 spaces for some θ. The fact that for a Banach function space F with non-trivial type each nuclear operator T on F for some $r < 2$ has r-th power summable eigenvalues is due to [Pi79, 3.6]. We do not know whether the class of all θ-type 2 spaces is a proper subclass of the class of all Banach spaces with non-trivial type. Note that Kalton in [Kal92] showed that if F has type p with $1 < p < 2$, then one cannot expect in general that F is a θ-type 2 space for $\theta := 2/p'$ (the "expected" θ), but maybe it is true for some smaller θ.

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